HYPER VERSION

Received: *May 16, 2001,* Accepted: *May 22, 2001*

D-brane dynamics and the quantum hall effect

Steven S. Gubser and Mukund Rangamani

Joseph Henry Laboratories, Princeton University
Princeton, NJ 08544, USA
E-mail: ssgubser@ebenezer.princeton.edu, rmukund@princeton.edu

Abstract: We study the recently proposed D-brane configuration [1] modeling the quantum Hall effect, focusing on the nature of the interactions between the charged particles. Our analysis indicates that the interaction is repulsive, which it should be for the ground state of the system to behave as a quantum Hall liquid. The strength of interactions varies inversely with the filling fraction, leading us to conclude that a Wigner crystal is the ground state at small ν. For larger rational ν (still less than unity), it is reasonable to expect a fractional quantum Hall ground state.

Keywords: Field Theories in Lower Dimensions, Chern-Simons Theories, M(atrix) Theories, Brane Dynamics in Gauge Theories.

JHEP05(2001)041

Contents

1. Introduction

In [1] it was conjectured that a specific assembly of D-branes and fundamental strings would have a low-energy dynamics similar to systems displaying the fractional quantum Hall effect (FQHE). Specifically, a D2-brane in the shape of $S^2 \times \mathbb{R}$ is placed around K flat D6-branes, so that the spatial directions of the different branes are orthogonal. The radius ρ transverse to the D6-brane is the one spatial direction perpendicular to both brane world-volumes. Because of a topological constraint, K fundamental strings stretch from the D6-branes to the D2-brane. The ends of these strings carry electric charge in the U(1) gauge theory on the D2-brane world-volume. A large number of D0-branes are bound to the D2-brane, representing strong magnetic flux in this U(1) gauge theory. See figure 1.

The string ends on the D2-brane are the "electrons" (and will hereafter be referred to as such), and the D0-branes are the flux quanta. The infrared dynamics is supposed to involve nearly rigid motion of the strings, and possibly a binding of D0-branes to the strings as a manifestation of binding flux quanta to electrons.[1] Clearly this system exhibits features in common with quantum Hall systems. However it is known that putative quantum Hall systems exhibit a variety of phases, including the Wigner crystal and stripe phases (see [2, 3] for pedagogical reviews and refer-

[1]D0-branes stuck to a D2-brane would ordinarily be "dissolved," even in their classical description, to produce a uniform D0-charge (or magnetic field) on the D2-brane. It is not clear to us that the binding of individual D0-branes to strings is to be taken literally as a stringy analog of the binding of flux quanta, which is better described as a change of variables than as a localization of the physical magnetic field. Optimistically, some appropriate change of variables in the D-brane setup would produce quasi-D0-branes which can change the statistics of the electrons but don't carry the magnetic field. We thank S. Sondhi for a discussion on this point.

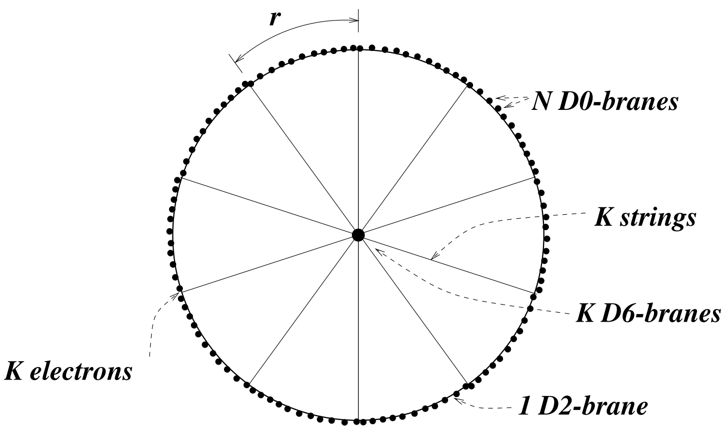

Figure 1: The "quantum Hall soliton" of [1]. The D6-branes are viewed end-on: they should be thought of as projecting out of the page in six orthogonal directions.

ences to the extensive condensed matter literature). It is the purpose of this note to inquire whether a fractional quantum Hall liquid is ever the ground state of the system. In general, this is a hard question which can be answered definitively only by diagonalizing the complete hamiltonian (including inter-electron interactions). We won't do this; but we will compute the force between electrons and find (modulo a plausible technical assumption) that it is repulsive. This is good because attractive interactions would inevitably lead to a clumping instability and no quantum Hall behavior. The characteristic energy scale of the interactions is comparable to the cyclotron gap, which is evidence that a quantum Hall liquid is at least not parametrically disadvantaged when compared to a Wigner crystal. Thus, our results lead us to be cautiously optimistic that a quantum Hall ground state exists at least for some filling fractions.

It is conceivable that some modification of the proposal of [1] would in fact parametrically favor a quantum Hall ground state in the thermodynamic limit, $N \to \infty$. The main desiderata are to weaken the inter-electron force and/or raise the cyclotron gap. If the goal is to have quantized transverse conductance, impurities are essential. For a clean subject like string theory, this may be the hardest part.

In section 2 we will briefly review some of the salient points of quantum Hall physics relevant to our analysis. In section 3 we will compute the inter-electron force and show that it is repulsive. This is a slightly delicate computation because the electrons almost enjoy a BPS no-force condition. Only finite volume effects break supersymmetry and thereby alter the BPS condition. The near-cancellation of inter-electron forces arising from scalar and gauge boson exchange can be viewed heuristically as an odd type of screening of the electrostatic repulsions which becomes more and more complete the closer the electrons get to one another. In section 4 we demonstrate that the repulsive interactions are of the same order of magnitude as the cyclotron gap.

While this paper was in preparation, [4] appeared, discussing possible instabilities of the D-brane system set up in [1]. It was shown that, if there is no binding of flux quanta, there are instabilities in the $\ell = 1$ and $\ell = 2$ partial waves on the $\mathbf{S}^2$; but if flux quanta do bind, there is no instability. This work is in a sense orthogonal to ours, since we focus on the inter-electron force and regard the binding of flux quanta as a derivative effect. Considerations similar to [4] may affect the stability of the Wigner crystal state toward long-wavelength fluctuations.

2. Some aspects of quantum Hall physics

Knowing the sign on the force between two string ends is important because it affects whether a quantum Hall state will form. The dynamical criteria for formation of quantum Hall states are, roughly,

1. There should be a repulsive force between charges.
2. The typical energy of these repulsive interactions should be less than the cyclotron gap, $\omega_c = eB/m_{\text{electron}}c$.
3. Charges should be crowded to within distances shorter than the uncertainties in their respective positions.

If 1. fails, then the charges tend to clump together. No quantum Hall state will form. The binding of flux quanta in real FQHE systems occurs to lower the repulsive Coulomb energy: the high power of $z_i - z_j$ in Laughlin's wave-function keeps the charges apart. Repulsive interactions between charges are the *sine qua non* of the fractional quantum Hall effect.

If 1. holds but 2. or 3. fail, then a Wigner crystal is generally preferred over the quantum Hall state. In fact, the Wigner crystal is a much more generic state of matter for variants of the two-dimensional electron gas. In [1], it was argued that a clean infrared limit existed where a quantum Hall ground state might be seen, but the arguments depended on having small filling fraction. In real two-dimensional electron gas systems with filling fraction below about $\nu = 1/7$, the conductance plateaux disappear and the ground state becomes a Wigner crystal (though without long-range order, since it's in two dimensions). We expect similar behavior in the D-brane system, but for a different reason than in real quantum Hall systems: as we will show in section 4, criterion 2. fails when ν is very small.

An issue which was left open in [1] is whether the electrons behave as fermions or bosons on the D2-brane. It seems most plausible that the electrons behave as bosons in their ground state: as a whole, the D2-D6 strings are fermions in their ground state, but the K string ends on the D6-branes need to be assembled into a gauge singlet of the SU(K) gauge theory with a $\epsilon_{i_1 \ldots i_K}$ tensor. As discussed in [1], an antisymmetric spatial wavefunction on the D6-brane world-volume would change the statistics of the electrons back to fermions. Bosons can form fractional quantum Hall states at even filling fractions: the Laughlin wave-function would involve even powers of $z_i - z_j$. Little of our analysis will rely on the statistics of the electrons in their ground state.

Finally, it is perhaps worth recalling the value of dirt in the quantum Hall effect. By "dirt" we mean quenched impurities. In the absence of dirt, one might invoke the Lorentz symmetry of the system to infer that the Hall conductance varies inversely with the magnetic field, the slope being proportional to the density of the electrons. At certain rational filling fractions one might still expect that the Laughlin wave-function is the ground state. Near such a ground state, the quasi-particle excitations will give rise to finite σ_{xx}, and σ_{xy} will vary with the filling fraction, so the characteristic plateaux will be absent from the conductance profile. When there is dirt, these quasi-particles localize, so $\sigma_{xx} = 0$ provided the Fermi level of the quasi-particle excitations lies within the energy gap and the number of quasi-particles is insufficient to drive the system into the next plateau. The transverse conductance, σ_{xy}, receives contributions only from boundary states, and is quantized. It is somewhat analogous to the θ-angle in QCD, and it is independent of bulk characteristics like the geometry of the sample. To summarize, in totally clean samples like the ones we will consider the transverse conductivity profile won't exhibit the familiar plateaux; but the quantum Hall ground state can still prevail at isolated filling fractions.

3. The inter-electron force

A natural description of the magnetic flux is to make the D2-brane gauge theory non-commutative. A second consequence of the flux is that it introduces a Chern-Simons interaction into the gauge theory. This is effectively a mass term for the photon. The gauge theory also includes scalars corresponding to the transverse fluctuations of the D2-brane. These scalars couple to the string ends. For radially directed fundamental strings, the scalar corresponding to radial fluctuations is the only one that couples to the string ends. This scalar is massive because the radius of the D0-D2 bound state is stable to spherical perturbations.

The force between two fundamental strings arises from two sources. First, there is an attractive force from bulk effects. Unless the strings run parallel (i.e. unless they are coincident), the attraction from graviton and dilaton exchange overcomes the repulsion from $B_{\mu\nu}$ exchange, because the strings are at angles. Second, there is a force from the dynamics of the D2-brane world volume theory. We will argue that this force is stronger in the large N limit. It is not so obvious *a priori* whether it is attractive or repulsive. The photon on the D2-brane world-volume induces a repulsive force, but the radial scalar induces an attractive force. The coupling constant for these two forces is the same, and they would cancel if it weren't for the effects of the D6-brane and the curvature of the D2-brane world-volume. So the question comes down to whether the photon or the scalar is more massive.

The total low-energy effective action on the D2-brane world-volume, in $+--$ signature, is

$$S_{\text{eff}} = \int d^3x\,\sqrt{g}\Big[J^0 A_0 - \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_\gamma \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \\ + (\text{fermions}) + (\text{interactions})\Big] + \sum_{i=1}^{K} Q\int_{\gamma_i}(A + \phi d\tau)\,, \quad (3.1)$$

where Q is the charge of a string end. There is also a uniform background charge J^0 which ensures overall charge neutrality. The terms in the second line of (3.1) indicate the couplings of the string ends to the gauge field and to the radial scalar ϕ. We have chosen to suppress the non-commutativity inherent in the action, for we shall mainly be concerned with calculating propagators and to this end modification of the action through introduction of star products is immaterial. The Chern-Simons and scalar masses are

$$\begin{aligned} m_\gamma &= \frac{1}{4}\frac{1}{(\pi N)^{1/3} l_s} \\ m_\phi &= \frac{4\sqrt{2}}{3}\frac{1}{(\pi N)^{1/3} l_s}\,, \end{aligned} \quad (3.2)$$

where $l_s = \sqrt{\alpha'}$. We will derive (3.2) explicitly in sections 3.1 and 3.2.

The remainder of this section will be devoted mainly to showing that $m_\gamma < m_\phi$ implies that there is a net repulsion between nearby electrons coming from D2-brane effects.

First consider a flat D2-brane with a finite density of D0-branes bound to it and perpendicular external strings attached, the same action would apply except with $m_\gamma = m_\phi = 0$ and $J^0 = 0$ (assuming no D6-branes). Such a system would be BPS, and the string ends would exert no force on one another. Bending the D2-brane into the shape of an S^2 breaks the supersymmetry, and we no longer expect a no-force condition to hold in the D2-brane world-volume theory.

Let us assume that the S^2 is large, and that K is also large, and ask what force there is between two fundamental string ends which are separated by a small angle. For this purpose it is enough to consider the quadratic part of the action S_{eff} in flat $\mathbb{R}^{2,1}$: we will compute only the tree-level propagators of the gauge field and the scalar. The theory is at weak coupling, so this should suffice to determine the force between string ends. The propagator for the scalar is $W(p) = i/(p^2 - m_\phi^2)$. To obtain the propagator for the gauge boson, we introduce a gauge fixing term and write the action as

$$
\begin{aligned}
S_{\text{gauge}} &= \int d^3x \left[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_\gamma \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right] \\
&= \int d^3x \, \frac{1}{2} A_\mu \left(m_\gamma \epsilon^{\mu\nu\rho}\partial_\nu + g^{\mu\rho}\Box\right) A_\rho \\
&\equiv \int d^3x \, \frac{1}{2} A_\mu S^{\mu\rho} A_\rho \,,
\end{aligned}
\tag{3.3}
$$

where in the second line we have gone to Feynman gauge, $\xi = 1$. Fourier transforming, one obtains

$$
S^{\mu\rho}(p) = -g^{\mu\rho}p^2 - i m_\gamma \epsilon^{\mu\nu\rho} p_\nu = \begin{pmatrix} -p^2 & i m_\gamma p_2 & -i m_\gamma p_1 \\ -i m_\gamma p_2 & p^2 & i m_\gamma p_0 \\ i m_\gamma p_1 & -i m_\gamma p_0 & p^2 \end{pmatrix} , \tag{3.4}
$$

where $\epsilon^{012} = 1$. It may seem peculiar to have imaginary components in $S^{\mu\rho}(p)$, but the i's are in the right places to make the minkowskian action real. We obtain the propagator by inverting:

$$
\begin{aligned}
W_{\mu\rho}(p) &= i\left[S^{\mu\rho}\right]^{-1} \\
&= \frac{i}{p^4(p^2 - m_\gamma^2)} \times \\
&\times \begin{pmatrix} -p^4 + p_0^2 m_\gamma^2 & i m_\gamma p_2 p^2 + p_0 p_1 m_\gamma^2 & -i m_\gamma p_1 p^2 + p_0 p_2 m_\gamma^2 \\ -i m_\gamma p_2 p^2 + p_0 p_1 m_\gamma^2 & p^4 + p_1^2 m_\gamma^2 & -i m_\gamma p_0 p^2 + p_1 p_2 m_\gamma^2 \\ i m_\gamma p_1 p^2 + p_0 p_2 m_\gamma^2 & i m_\gamma p_0 p^2 + p_1 p_2 m_\gamma^2 & p^4 + p_2^2 m_\gamma^2 \end{pmatrix} .
\end{aligned}
\tag{3.5}
$$

The potential energy arising from gauge boson exchange between two stationary string ends is obtained by differentiating the Fourier transform of $W_{00}(p)$ with p entirely spatial. The scalar mediated potential energy is obtained similarly from $W(p)$. For entirely spatial p,

$$
\begin{aligned}
W_{00}(p) &= \frac{i}{\vec{p}^2 + m_\gamma^2} \\
W(p) &= -\frac{i}{\vec{p}^2 + m_\phi^2} \,.
\end{aligned}
\tag{3.6}
$$

Thus we see that the potential energies from gauge bosons and from scalars have the same functional form, up to a sign. The repulsion due to the gauge bosons dominates when $m_\gamma < m_\phi$. Some subtleties on the normalization of the potential will be discussed in section 3.1. In section 3.3 we will argue that the bulk contribution is negligible.

The only other ingredient necessary to compute the force is the strength of the coupling between the gauge field and the electrons. In section 3.1 we shall show that

JHEP05(2001)041

the electron couples with a strength q given by

$$q = \sqrt{\frac{1}{2\nu l_s}\left(\frac{\pi^2}{N}\right)^{1/3}}. \tag{3.7}$$

The momentum space potential contributed by the gauge field is

$$V_\gamma(\vec{p}) = q^2 W_{00}(\vec{p}) = q^2 \frac{i}{\vec{p}^2 + m_\gamma^2}. \tag{3.8}$$

Fourier transforming back to position space gives

$$V_\gamma(r) = 2\pi q^2 K_0(m_\gamma r). \tag{3.9}$$

Recalling that the scalar contribution has the same functional form as V_γ, and that the net force should vanish in the $r \to 0$ limit because the BPS property is asymptotically recovered at short distances, we conclude that the total tree-level potential from the D2-brane gauge theory is

$$V_{\mathrm{bdy}}(r) = 2\pi q^2 \left[K_0(m_\gamma r) - K_0(m_\phi r)\right]. \tag{3.10}$$

The potential scales with N and ν like $q^2 \sim \frac{1}{\nu N^{1/3}}$.

3.1 The gauge field

In [1], the effect of six-branes in the set-up was modeled by replacing them by their gravitational background, which we reproduce here for convenience. The spacetime metric (in rescaled coordinates) for K D6-branes is

$$ds^2 = \sqrt{\frac{\rho}{l_s}}\left(d\tau^2 - d\tilde{y}^a d\tilde{y}^a\right) - \sqrt{\frac{l_s}{\rho}}\left(d\rho^2 + \rho^2 d\Omega_2^2\right) \tag{3.11}$$

and the background dilaton is

$$g_s^2 e^{2\Phi} = \frac{4}{K^2}\left(\frac{\rho}{l_s}\right)^{3/2}. \tag{3.12}$$

It was also shown that a spherical D2-brane with N-units of magnetic flux is stable at a radius given by

$$\rho_* = \frac{(\pi N)^{2/3}}{2}\, l_s. \tag{3.13}$$

Since the gauge theory lives on the world-volume of the D2-brane, we can infer that the Yang-Mills coupling of the theory is given as

$$g_{YM}^2 l_s = g_s e^{\Phi}|_{\rho_*} = 2^{\frac{1}{4}}\frac{\sqrt{\pi N}}{K}. \tag{3.14}$$

JHEP05(2001)041

The authors of [1] give an open string metric, eq. (5.15) to be precise, which is computed using the standard Seiberg-Witten prescription [5] using a flat closed string metric and a B-field of appropriate strength. The D2-brane unfortunately does not live in flat space. The induced metric in the static gauge is

$$ds^2_{\text{ind}} = \frac{(\pi N)^{1/3}}{\sqrt{2}} d\tau^2 - \frac{(\pi N)}{2\sqrt{2}} l_s^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,. \tag{3.15}$$

Using the B-field to be $B_{\theta\phi} = \frac{N}{2}\sin\theta$, we can evaluate the correct open string metric as seen by the D2-brane world volume theory as

$$ds^2_{\text{open}} = \frac{(\pi N)^{1/3}}{\sqrt{2}} \left(d\tau^2 - \frac{9 l_s^2}{2} (\pi N)^{2/3} (d\theta^2 + \sin^2\theta \, d\phi^2) \right) \,. \tag{3.16}$$

It is this metric that appears in the non-commutative gauge boson kinetic term. So to get the right normalizations for the gauge bosons, start from the action

$$S_{\text{gauge}} = -\frac{1}{2g^2_{YM}} \int d^3\xi \sqrt{G_{\text{open}}} \, G^{\mu\rho}_{\text{open}} G^{\nu\sigma}_{\text{open}} F_{\mu\nu} F_{\rho\sigma} \,. \tag{3.17}$$

To ensure that we write the scalar and the gauge boson action in terms of the same time coordinate, let us conformally rescale the metric by writing the action in terms of a metric we call $\tilde{G}_{\text{open}}$. Since we would like to put our action in canonical form as in (3.1) we would need to rescale the gauge fields to achieve this end. Defining $\alpha = \frac{(\pi N)^{1/3}}{\sqrt{2}}$, we want $\alpha \tilde{G}_{\text{open}} = G_{\text{open}}$, implying $\sqrt{G_{\text{open}}} \, G^{\mu\rho}_{\text{open}} G^{\nu\sigma}_{\text{open}} = \frac{1}{\sqrt{\alpha}} \sqrt{\tilde{G}_{\text{open}}} \, \tilde{G}^{\mu\rho}_{\text{open}} \tilde{G}^{\nu\sigma}_{\text{open}}$. Hence we can cast the gauge boson kinetic term in the canonical form by defining, $A = \alpha^{1/4} \sqrt{\frac{g^2_{YM}}{2}} \tilde{A}$. The gauge boson mass term is

$$\begin{aligned} \frac{K}{4\pi N} \int d^3\xi \epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma &= \frac{K}{4\pi N} \int d^3\xi \sqrt{\alpha} \frac{g^2_{YM}}{2} \, \epsilon^{\mu\nu\sigma} \, \tilde{A}_\mu \partial_\nu \tilde{A}_\sigma \\ &= \int d^3\xi \, \frac{m_\gamma}{2} \, \epsilon^{\mu\nu\sigma} \tilde{A}_\mu \partial_\nu \tilde{A}_\sigma \end{aligned} \tag{3.18}$$

with

$$m_\gamma = \frac{K}{4\pi N} g^2_{YM} \sqrt{\alpha} = \frac{1}{4} \frac{1}{(\pi N)^{1/3} l_s} \,.$$

In the above series of manipulations we have taken cognizance of the fact that the Chern-Simons term is topological and hence will remain unaffected by the conformal rescaling of the metric. Note that the Compton wavelength of the photon is indeed of the same order as the size of the sphere measured in units prescribed by the metric $\tilde{G}_{\text{open}}$.

One other ingredient that will be necessary is a proper normalization of the coupling of the "electrons" to the gauge field. This normalization can be fixed by comparing the coupling to the chemical potential term, for the system is constrained to have exactly K "electrons". Writing the action in terms of the rescaled variables introduced above, we have

$$S_{\text{coupling}} = \sum_{i=1}^{K} \frac{1}{V} \int d^3\xi \alpha^{1/4} \sqrt{\frac{g_{YM}^2}{2}} \tilde{A} + \int d^3\xi \; \tilde{A}_0 \tilde{J}^0 \tag{3.19}$$

with $\tilde{J}^0 = \frac{K}{V} \alpha^{1/4} \sqrt{g_{YM}^2/2}$. Varying the above with respect to $\tilde{A}$ we see that the chemical potential term is saturated by the presence of K electrons. The main point of this is that an electron couples to the gauge field $\tilde{A}$ with strength $q = \alpha^{1/4} \sqrt{g_{YM}^2/2}$, as promised in (3.7).

3.2 The scalar action

The DBI action for the 2-brane (treated as a probe) in the near-horizon geometry of the D6-branes was used to compute the potential of the radial mode scalar, and to show that there is indeed a radius wherein the D2-D0 bound state could be stabilized. Indeed the same approach can be extended to compute the scalar kinetic terms, a necessary ingredient in determining the mass of the radial mode.

Choosing to work in static gauge with coordinates

$$\xi^0 = \tau, \;\; \xi^1 = \theta, \;\; \xi^2 = \phi \,, \tag{3.20}$$

the induced metric, given the spacetime metric (3.11), turns out to be

$$\begin{aligned} G_{00} &= \sqrt{\frac{\rho}{l_s}} - \sqrt{\frac{l_s}{\rho}} \left(\frac{\partial\rho}{\partial\tau}\right)^2 \\ G_{11} &= -\sqrt{\rho^3 l_s} - \sqrt{\frac{l_s}{\rho}} \left(\frac{\partial\rho}{\partial\theta}\right)^2 \\ G_{22} &= \sqrt{\rho^3 l_s} \sin^2\theta - \sqrt{\frac{l_s}{\rho}} \left(\frac{\partial\rho}{\partial\phi}\right)^2 \,. \end{aligned} \tag{3.21}$$

In addition we have the world-volume field strength turned on,

$$F_{12} = \frac{N}{2} \sin\theta \,. \tag{3.22}$$

Plug all of this into the DBI action:

$$\begin{aligned} L_{DBI} &= -\frac{1}{4\pi^2 g_s l_s^3} \int d\tau d\theta d\phi \; e^{-\Phi} \det[G_{ab} + 2\pi l_s^2 F_{ab}]^{\frac{1}{2}} \qquad (3.23) \\ &= \frac{K}{8\pi^2 l_s^2} \int d^3\xi \rho \sqrt{1+\frac{N^2\pi^2 l_s^3}{\rho^3}} \left\{ -1+\frac{1}{2}\frac{l_s}{\rho}\left(\frac{\partial\rho}{\partial\tau}\right)^2 - \frac{1}{2}\left(\nabla_T \rho\right)^2 \left(\frac{1}{\rho^2}\frac{1}{1+\frac{N^2\pi^2 l_s^3}{\rho^3}}\right)\right\} . \end{aligned}$$

The first term in the action (the -1 part) is just the potential term that was evaluated in [1]. Expanding about the critical point $\rho = \rho_*$ we get

$$\begin{aligned} S_{\text{scalar}} &= \int d^3\xi \left\{ \frac{3K}{8\pi^2 l_s} \left[\frac{1}{2}\left(\frac{\partial\rho}{\partial\tau}\right)^2 - \frac{2}{9l_s^2(\pi N)^{2/3}}\frac{1}{2}\left(\nabla_T\rho\right)^2 \right] - \frac{2}{3\pi^2 l_s^3}\frac{K}{(N\pi)^{2/3}}\rho^2 \right\} \\ &= \int d^3\xi \sqrt{\tilde{G}_{\text{open}}} \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\frac{2}{9l_s^2(\pi N)^{2/3}}\left(\nabla_T\phi\right)^2 - \frac{16}{9l_s^2}\frac{1}{(\pi N)^{2/3}}\phi^2 \right]. \end{aligned} \tag{3.24}$$

We have rescaled the scalar ρ and written the action in terms of a new scalar ϕ, so that the metric seen by the scalars is also $\tilde{G}_{\text{open}}$. This gives

$$m_\phi^2 = \frac{32}{9l_s^2}\frac{1}{(\pi N)^{2/3}}\,.$$

Hence we find that m_ϕ and m_γ are of the same order in N, but that the scalar is heavier by a numerical factor of $16\sqrt{2}/3$.

3.3 Estimating the bulk force

It is difficult to compute the force between a pair of strings due to the exchange of massless string modes: the strings are finite in length, and the background is non-trivial. However, by making some reasonable assumptions, we can estimate the potential arising from closed string exchange. Let ϑ be the angle between a pair of strings. Our first assumption is that the potential has the form

$$V_{\text{bulk}}(\vartheta) = V_0(1-\cos\vartheta)\,. \tag{3.25}$$

While this may not be exactly right, it seems very likely to be close enough for our purposes. More specifically, the property which we expect the true $V_{\text{bulk}}(\vartheta)$ to share with (3.25) is that it has only one scale: the maximum value of $V_{\text{bulk}}(\vartheta)$ is of the same order of magnitude as the second derivative at $\vartheta = 0$ (this is the "plausible technical assumption" mentioned in the introduction). A mild singularity at $\vartheta = 0$, such as a $\vartheta^2\log\vartheta$ term, would not affect our conclusions. A sharper singularity at $\vartheta = 0$ would be unexpected since $\vartheta = 0$ is where a no-force condition is restored.

Second, we assume that V_0 may be estimated as the magnitude of the gravitational potential energy experienced by two point masses in flat ten-dimensional space, separated by the same distance as the endpoints of the strings at angle $\vartheta = \pi/2$, and having the same mass as the strings. This assumption is safe as long as there isn't a strong force coming from the region very close to the D6-brane. The mass of the strings is[2]

$$m_{\text{str}} = \frac{1}{2\pi l_s}\frac{\rho_*}{l_s} \sim \frac{N^{2/3}}{l_s} \tag{3.26}$$

[2] We are defining mass by integrating the Nambu-Goto action of the string over the spatial coordinate: $S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{\det G_{\mu\nu}} = -m_{\text{str}}\int d\tau$ for a static string. This results in a slightly different answer from [1], but we believe our approach is the correct one for our purposes.

and they are separated by a distance

$$L \sim \rho_* \left(\frac{l_s}{\rho_*}\right)^{1/4} \sim l_s \sqrt{N} \,. \tag{3.27}$$

Newton's constant is $G_N = g_s^2 e^{2\phi}\big|_{\rho_*} \sim l_s^8 N/K^2$. The gravitational potential energy has magnitude

$$V_0 = \frac{G_N m_{\text{str}}^2}{L^7} \sim \frac{1}{l_s} \frac{1}{\nu^2 N^{19/6}} \,. \tag{3.28}$$

Clearly, in a $N \to \infty$ limit with ν held fixed, V_0 scales to zero much faster than the magnitude of the potential induced by D2-brane effects (see the text following (3.10)). Thus we are justified in asserting that bulk effects are negligible. This is gratifying because it verifies that we are working in a decoupling limit, where gravitational effects are much weaker than the open string effects on the D2-brane world-volume.

4. Quantum Hall fluid or Wigner crystal?

In general it is difficult to be sure whether a particular hamiltonian will or won't lead to quantum Hall behavior without performing some diagonalization or robust variational calculation. However a figure of merit which serves as a useful guide to the physics is a typical energy of interactions divided by the cyclotron gap,

$$\eta = \frac{V_{\text{typ}}}{\omega_c} \,. \tag{4.1}$$

Physically, the smallness of this ratio is a measure of the validity of projecting the system to the lowest Landau level and treating the interactions perturbatively. If η is small, a quantum Hall ground state is clearly favored. If η is very large, then there is no reason for the ground state to be close to a combination of lowest Landau level wave-functions; instead one may expect a Wigner crystal to win out energetically.

To compute the cyclotron gap, one should in principle start from the string worldsheet and compute the mode of excitation corresponding to cyclotron motion.[3] However, because this is the lowest excitation mode available to the string, it is a fair approximation to say that the string moves rigidly. Thus it suffices, at least for the purposes of estimating ω_c up to factors of order unity, to keep track only of the dynamics of the end of the string. For this purpose we need the action

$$S = -m \int ds + q \int \tilde{A} \,. \tag{4.2}$$

[3]We thank L. Susskind and N. Toumbas for pointing out to us that our original estimate of ω_c was parametrically smaller than the correct answer, leading us to the incorrect conclusion that the Wigner crystal was favored over the quantum Hall state. An approximate worldsheet calculation was also supplied to us by L. Susskind, which leads to an answer similar to the computation presented here.

JHEP05(2001)041

The line element ds is defined by $ds^2 = \tilde{G}^{\rm open}_{\mu\nu} d\xi^\mu d\xi^\nu$, so that for a static string, $ds = d\tau$. The mass m is $m_{\rm str}$ computed in (3.26). Let us assume that the string end is near the equator, $\theta = \pi/2$, of the S^2, and choose local coordinates x_1 and x_2 for the position of the string end such that $dx_1 = d\theta$ and $dx_2 = d\phi$. Then, setting $l_s = 1$ and dropping up to factors of order unity, $q\tilde{F}_{12} = QF_{12} \sim N$, $m \sim N^{2/3}$, and $ds = d\tau\sqrt{1 - N^{2/3}\dot{x}_\alpha^2}$. Making the non-relativistic approximation where the square root in the last expression for ds is expanded out to leading order, one finds from (4.2) the equations for cyclotron motion with

$$\omega_c = \frac{QF_{12}}{N^{2/3}m} \sim N^{-1/3} \,. \tag{4.3}$$

In real quantum Hall systems, typical interaction energies are computed as e^2/ℓ where ℓ is the average separation between nearest neighbors. Here the potential behaves as $q^2 r^2 \log m_\gamma r$ for small separations r, so we might conclude that interactions are very weak indeed, and that a quantum Hall ground state is favored. This is overly optimistic, since the forces between non-nearest-neighbors have not been accounted for. To obtain a more conservative estimate of the typical interaction energy, let us assume that the electrons are approximately evenly spaced on the sphere, and see what energy it would take to move one electron to the location of its nearest neighbor. The magnitude of this energy may be estimated as $V_{\rm typ} \sim q^2 \sim 1/(\nu N^{1/3})$ in string units. (One way to get at this is to replace the sphere by a circular patch with a uniform density of electrons, and then compare the potential energy of an electron at the center of the circle to one slightly displaced from the center). Thus the figure of merit turns out to be $\eta \sim 1/\nu$.

Because we find no parametric dependence of η on N, we cannot with confidence claim that the ground state at moderate values of ν will be a quantum Hall liquid or a Wigner crystal in the thermodynamic limit where $N \to \infty$. Real quantum Hall systems in fact have η considerably larger than 1. In fact, as remarked in section 2, such systems have a Wigner crystal phase for small ν ($\nu \lesssim 1/7$), a quantum Hall phase at slightly higher ν (for instance, $\nu \approx 1/3$), and yet other phases, like stripes, at larger ν (like $\nu \approx 9/2$). The D-brane system should exhibit a Wigner crystal phase at small ν, though for different reasons than real quantum Hall systems: for the D-brane system, $\eta \sim 1/\nu$, so the repulsive interactions become strong as $\nu \to 0$. This seems backwards in comparison to real quantum Hall systems, where $\nu \to 0$ corresponds to extremely strong magnetic field. The D-brane system is different in that the magnetic flux per unit area is essentially constant. The S^2 adjusts its size to make it so. Small ν pushes up the gauge coupling, and with it the strength of the repulsive interactions.

On the other hand, if ν is too large, then string excitations are energetically available which reverse the statistics of the electrons. Different excited string states should be viewed as separate species of particles, but with the same electric charge. It's not clear what the ground state will be in this case. Stripes with integer filling

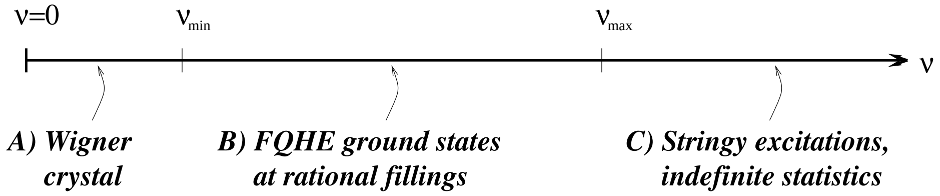

Figure 2: Expected "phase diagram" of the quantum Hall soliton. It would take numerics to figure out the value of $\nu_{\rm min}$. If the electrons are fermions in their ground state, a reasonable guess is $\nu_{\rm max} = 1$. If they are bosons, multiple occupancy of lowest Landau level states may push $\nu_{\rm max}$ somewhat higher.

fractions of each species is perhaps a reasonable guess—but the system is complicated, as can be seen from the breakdown of the arguments in [1] that quasi-particle energies are smaller than other energy scales in the system.

For an intermediate range of ν, where interactions are under reasonable control but only the lowest string mode is available, one may hope that a conventional quantum Hall liquid is indeed the ground state of the system (see figure 2). This expectation should be born out by numerics, comparing a Laughlin-like wavefunction to a Wigner crystal for various values of ν. It would be nice to contrive a D-brane set-up where interactions can be made parametrically weak, so that fractional quantum states are clear winners over any other state of the system for some range of ν. To do this, one might for instance try to lower the mass of the electrons (and thereby raise the cyclotron gap) by having the strings end not on the D6-branes but on some other brane closer to the D2-brane. An anti-D2-brane concentric with the D2-brane might approximately fit the bill. Precisely this possibility was discussed in [1], and it was found that there was no energy barrier toward creating such an anti-D2-brane in the strict near-horizon limit for the D6-brane; but restoring the one to the harmonic function throws up a slight potential gradient preventing it. On top of this there is a brane-anti-brane attraction. Although our investigation has not been detailed, it seems that the only stable (or meta-stable) equilibrium point is the one with no anti-D2-brane: in particular, the "phenomenologically" attractive configuration where the anti-D2-brane sits very close to the D2-D0 bound state appears to be unstable. More elaborate brane configurations worthy of consideration include intersecting or nearly-intersecting branes with the electrons arising from short strings between two nearby branes.

JHEP05(2001)041

Acknowledgments

We would like to thank J. Maldacena, I. Klebanov, L. Susskind, N. Toumbas, A. Vishwanath and especially S. Sondhi for useful discussions. The work of S.S.G. was supported in part by DOE grant DE-FG02-91ER40671, and by a DOE Outstanding Junior Investigator award.

References

[1] J.H. Brodie, L. Susskind and N. Toumbas, *How Bob Laughlin tamed the giant graviton from Taub-NUT space*, *J. High Energy Phys.* **02** (2001) 003 [hep-th/0010105].

[2] A. Karlhede, S.A. Kivelson, and S.L. Sondhi, *The quantum hall effect: the article*, lecture notes from the 9th Jerusalem Winter School, 1992.

[3] S.M. Girvin, *The quantum Hall effect: novel excitations and broken symmetries*, cond-mat/9907002.

[4] I. Bena and A. Nudelman, *On the stability of the quantum hall soliton*, *J. High Energy Phys.* **12** (2000) 017 [hep-th/0011155].

[5] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].